COMPLEX ANALYSIS
TOPIC IX: THE COMPLEX EXPONENTIAL FUNCTION

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1. Exponentiation

We have yet to thoroughly investigate the meaning of exponentiation of complex numbers; that is, how do we define a^z, where $a, z \in \mathbb{C}$?

1.1. **When z is an integer.** At this point, we can confidently state that the first three properties we obtain for a^x when a is real and $n = x$ is an integer are carried over to the case of complex base, and for the same reasons.

If $n \in \mathbb{N}$, we define

- **(a)** a^n means a multiplied with itself n times;
- **(b)** $a^0 = 1$;
- **(c)** $a^{-n} = \dfrac{1}{a^n}$.

1.2. **When z is rational.** Consider the real function $f : [0, \infty) \to [0, \infty)$ given by $f(x) = x^n$, where n is a positive integer with $n \geq 2$. This function is bijective, with inverse function $f^{-1}(x) = \sqrt[n]{x}$. So, when a is real and positive, there exists a unique positive $b \in \mathbb{R}$ such that $b^n = a$, and we define $\sqrt[n]{a} = b$.

The situation is not so clear for $a \in \mathbb{C}$. If a is complex and nonzero, we know a has n distinct n^{th} roots. In order to define a function and n^{th} root function, we must choose a preferred n^{th} root in a consistent manner. This can be done by taking the n^{th} root with the smallest positive angle among all the n^{th} roots.

Let $z \in \mathbb{C}$, and set $r = |z|$ and $\theta = \operatorname{Arg}(z)$ if $z \neq 0$, and $\theta = 0$ if $z = 0$. Recall that $\operatorname{Arg}(z) \in (-\pi, \pi]$, so that $z = r \operatorname{cis} \theta$ in a unique way. Define

$$\sqrt[n]{z} = \sqrt[n]{r} \operatorname{cis}\left(\frac{\theta}{n}\right).$$

Set $D = \{z \in \mathbb{C} \mid 0 \leq \operatorname{Arg}(z) < \frac{2\pi}{n}\}$. The function

$$f : D \to \mathbb{C} \quad \text{given by} \quad f(z) = z^n$$

is bijective, with inverse

$$g : \mathbb{C} \to D \quad \text{given by} \quad g(z) = \sqrt[n]{z}.$$

Getting back to our question, "what is a^z", we define this in the case that $z = \dfrac{m}{n}$, where $m, n \in \mathbb{Z}$, by

$$a^z = a^{m/n} = \sqrt[n]{a^m}.$$

For $n = 2$, we note that anything of the form $\pm\sqrt[2]{a}$ is a square root of a. However, for larger n, anything of the form $\operatorname{cis}\left(\dfrac{2\pi k}{n}\right)\sqrt[n]{a}$ is an n^{th} root of a.

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2. Definition of the Exponential Function

Recall the function

$$\operatorname{cis} : \mathbb{R} \to \mathbb{C} \quad \text{given by} \quad \operatorname{cis}(\theta) = \cos\theta + i\sin\theta.$$

Using power series, we gave a justification for Churchill's use of the notation $e^{i\theta}$ to mean $\operatorname{cis}\theta$. If we wish to define e^z, where $z = x + iy$, we use a standard property of exponentiation to write

$$e^{x+iy} = e^x e^{iy} = e^x \operatorname{cis} y.$$

This motivates the following definition. Let $\mathbb{C}^* = \mathbb{C} \smallsetminus \{0\}$.

Definition 1. Define the *(complex) exponential function* by

$$\exp : \mathbb{C} \to \mathbb{C}^* \quad \text{by} \quad \exp(x + iy) = e^x \operatorname{cis}(y).$$

Proposition 1. *The domain of* exp *is* $\mathbb{C}$*, and the range is* $\mathbb{C}^*$*.*

Proof. Clearly, any $z = x + iy$ can be plugged into the definition $\exp(z) = e^x \operatorname{cis} y$, so the domain of exp is $\mathbb{C}$. Since $|\exp(z)| = e^x > 0$, $\exp(z)$ is never zero, for any z. However, if $w \neq 0$, then $w = r\operatorname{cis}\theta$ for some $r, \theta \in (0, \infty)$. Set $x = \log(r)$ and $y = \theta$ to see that $\exp(z) = e^{\log r}\operatorname{cis}(y) = r\operatorname{cis}\theta = w$. Thus exp is into and onto $\mathbb{C}^* = \mathbb{C} \smallsetminus \{0\}$. □

For purely real z, we have $z = x + iy$ with $y = 0$, so $\exp(z) = e^x \operatorname{cis} 0 = e^x$; thus exp agrees with the real exponential function for real values of z.

The definition of exp appears in polar form in the range. In rectangular form,

$$\exp(x + iy) = u(x, y) + iv(x, y),$$

where

$$u(x, y) = e^x \cos y \quad \text{and} \quad v(x, y) = e^x \sin y.$$

Proposition 2. *The function* exp *is entire, with*

$$\frac{d}{dz}\exp(z) = \exp(z).$$

Proof. We apply the converse of the Cauchy Riemann equations.

$$\begin{aligned} u_x &= e^x \cos y & \qquad v_x &= e^x \sin y \\ u_y &= -e^x \sin y & \qquad v_y &= e^x \cos y \end{aligned}$$

The partials are continuous, and the Cauchy Riemann equations are satisfied. Thus the derivative exists are each point $z = x + iy \in \mathbb{C}$, and

$$\frac{d}{dz}\exp(z) = u_x(x, y) + iv_x(x, y) = e^x \cos y + ie^x \sin y = e^x \operatorname{cis} y = \exp(z).$$

□

3. Algebraic Properties of the Exponential Function

Proposition 3. *Let $z_1, z_2 \in \mathbb{C}$. Then*

$$\exp(z_1 + z_2) = \exp(z_1) \cdot \exp(z_2).$$

Proof. Let $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$. We have

$$\begin{aligned}\exp(z_1 + z_2) &= \exp((x_1 + x_2) + i(y_1 + y_2)) \\ &= e^{x_1+x_2} \operatorname{cis}(y_1 + y_2) \\ &= e^{x_1} e^{x_2} \operatorname{cis}(y_1) \operatorname{cis}(y_2) \\ &= e^{x_1} \operatorname{cis}(y_1) e^{x_2} \operatorname{cis}(y_2) \\ &= \exp(z_1) \exp(z_2).\end{aligned}$$

□

Proposition 4. *Let $z \in \mathbb{C}$ and $n \in \mathbb{N}$. Then*

$$(\exp(z))^n = \exp(nz).$$

Reason. This follow from Proposition 3 and induction. □

The reader who has experience with the concept of groups will see that

$$\exp : \mathbb{C} \to \mathbb{C}^*$$

is a group homomorphism, from the additive group of complex numbers, to the multiplication group of nonzero complex numbers.

When restricted to the reals,

$$\exp : \mathbb{R} \to (0, \infty)$$

is a group isomorphism from the additive group of real numbers to the multiplicative group of positive real numbers; this is because exp is injective on the reals. The inverse function is the natural logarithm,

$$\log : (0, \infty) \to \mathbb{R}.$$

However, the complex exponential function is far from injective; indeed, every point in the range has infinitely many preimages. It is this fact that makes complex logarithms a much deeper topic than it had been for the reals.

4. Mapping Properties of the Exponential Function

We investigate the mapping properties of the exponential function.

Proposition 5. *The exponential function satisfies these properties.*

- **(a)** $\lim_{\operatorname{Re} z \to +\infty} \exp(z) = \infty$;
- **(b)** $\lim_{\operatorname{Re} z \to -\infty} \exp(z) = 0$.

Proof. First, we note that, for $z = x + iy$,

$$|\exp(z)| = |\exp(x+iy)| = |e^x \operatorname{cis}(y)| = |e^x||\operatorname{cis}(y)| = |e^x| \cdot 1 = e^x.$$

For **(a)**, we understand the meaning to be that as the real part of z tends to $+\infty$, the magnitude of $\exp(z)$ also tends to $+\infty$. But this is certainly the case, since

$$\lim_{x \to +\infty} |\exp(x+iy)| = \lim_{x \to +\infty} e^x = +\infty.$$

For **(b)**, we understand the meaning to be that as the real part of z tends to $-\infty$, the magnitude of $\exp(z)$ also tends to 0. But this is also the case, since

$$\lim_{x \to -\infty} |\exp(x+iy)| = \lim_{x \to -\infty} e^x = 0.$$

□

Definition 2. A complex function $f : D \subset \mathbb{C} \to \mathbb{C}$ is *periodic* if there exists a complex number $T \in \mathbb{C}$, with $T \neq 0$, such that

$$f(z+T) = f(z)$$

for all $z \in D$. We say that T is a *period* of the periodic function if $f(z+T) = f(z)$ for all D in D, and is not the case that $f(z+aT) = f(z)$ for all D in D, for any $0 < a < 1$.

Proposition 6. *The exponential function satisfies these properties.*

- **(a)** exp *is periodic with period* $2\pi i$;
- **(b)** exp *sends horizontal lines to rays from the origin;*
- **(c)** exp *sends vertical lines to circles centered at the origin.*

Proof. For **(a)**, note that

$$\exp(z + 2\pi i) = \exp(x + i(y+2\pi)) = e^x \operatorname{cis}(y+2\pi) = e^x \operatorname{cis} y.$$

Moreover, it is clear that $2\pi i$ is the smallest multiple of i for which this is true.

For **(b)**, let L be the horizontal line with equation $y = \phi$, where ϕ is fixed. The image of this line is the set with equation $e^x \operatorname{cis} \phi$. Since e^x is increasing and its image is $(0, \infty)$, we see that this equation has the same locus as the equation $\operatorname{Arg} z = \phi$, which is a ray with angle ϕ which excludes the origin. The mapping of each line is injective, but each ray is the image of an infinite number of lines.

For **(c)**, let L be a vertical line with equation $x = \rho$, where ρ is fixed. The image of this line is the set with equation $e^\rho \operatorname{cis} y$. The locus of this equation is a circle, centered at the origin, with radius e^ρ. Each circle is the image of exactly one vertical line, but each point on a circle has an infinite preimage on the line. □

5. The Principle Logarithm

Recall that we have indicated that $\arg z$ can be taken to mean any angle θ, $\arg z = \theta$, such that $z = |z| \operatorname{cis} \theta$. Then $\arg z$ is not a well-defined function. To make a function, we defined the *principal argument* to be a function

$$\operatorname{Arg} : \mathbb{C}^* \to \mathbb{R} \quad \text{defined by} \quad \operatorname{Arg} z = \theta \in (-\pi, \pi] \text{ with } z = |z| \operatorname{cis} \theta.$$

Similarly, we wish to let $\log z$ denote any complex number w with the property that $\exp(w) = z$. In order to make a function of this, we must restrict the domain of exp to one on which exp is injective. For the rest of this section, if $\rho \in \mathbb{R}$, let $\operatorname{Log} \rho$ denote the (real) natural logarithm of ρ.

Let $D = \{z \in \mathbb{C} \mid -\pi < \operatorname{Im}(z) < \pi\}$ and let $\mathbb{C}^+ = \mathbb{C} \setminus (-\infty, 0]$. Restriction gives a function

$$\operatorname{Exp} : D \to \mathbb{C}^+ \quad \text{given by} \quad \operatorname{Exp}(z) = \exp(z).$$

Then Exp is bijective, and so it has an inverse function. To investigate this, let $w \in D$ and suppose $\operatorname{Exp}(w) = z$, where $w = u + iv$. Then

$$|z| \operatorname{cis}(\operatorname{Arg}(z)) = z = \operatorname{Exp}(w) = e^u \operatorname{cis} v,$$

we have $|z| = e^u$, so that $u = \operatorname{Log}(|z|)$, and $v = \operatorname{Arg}(z)$.

Define the *principal logarithm* to be the function

$$\operatorname{Log} : \mathbb{C}^+ \to D \quad \text{given by } \operatorname{Log}(z) = \operatorname{Log}|z| + i \operatorname{Arg}(z).$$

Then

$$\operatorname{Exp}(\operatorname{Log}(z)) = \operatorname{Exp}(\operatorname{Log}|z| + i \operatorname{Arg}(z)) = e^{\operatorname{Log}|z|} \operatorname{cis}(\operatorname{Arg}(z)) = |z| \operatorname{cis}(\operatorname{Arg}(z)) = z.$$

Similarly,

$$\operatorname{Log}(\operatorname{Exp}(w)) = \operatorname{Log}(e^{\operatorname{Re} w} \operatorname{cis}(\operatorname{Im} w)) = \operatorname{Log}|e^{\operatorname{Re} w}| + i \operatorname{Im} w = \operatorname{Re} w + i \operatorname{Im} w = w.$$

Thus, Log is the inverse of Exp.

We note that it is impossible to define Log on $\mathbb{C}^*$ such that Log is continuous; for suppose that for z a negative real number, we defined $\operatorname{Log}(z) = \operatorname{Log}|z| + i\pi$. Then, Log is not continuous at z; every neighborhood of z contains points with negative imaginary parts, whose principle argument is negative, and thus not near π. We prefer to take Log as an function which is differentiable in its open domain.

6. Branches of Argument and Logarithm

Recall that we have indicated that $\arg z$ can be taken to mean any angle θ, $\arg z = \theta$, such that $z = |z| \operatorname{cis} \theta$. This leaves us in a state such that $\arg z$ is not a well-defined function. Similarly, we wish to let $\log z$ denote any complex number w with the property that $\exp(w) = z$, but this "function" is again not well-defined, in that there is more than one complex value it could represent. In order to create well-defined functions, we established the principle values of argument and logarithm. However, there are occasions where the other possible values should also be considered; we wish to formalize this situation of "multi-valued functions". We could take the approach that log is a function which takes values in the power set of $\mathbb{C}$, by setting $\log(z) = \{w \in \mathbb{C} \mid \exp(w) = z\}$, so that $\log(z)$ is the fiber over z under the exponential map. However, the notion of branches suits us better.

Definition 3. Let $A \subset \mathbb{C}$. We say that A is *disconnected* if there sets $U, V \subset \mathbb{C}$ such that

- **(a)** U and V are open;
- **(b)** $A \cap U \neq \varnothing$ and $A \cap V \neq \varnothing$;
- **(c)** $U \cap V = \varnothing$.

We say that A is *connected* if it is not disconnected.

Definition 4. Let D be an open connected subset of $\mathbb{C}$.

A *branch of argument* on D is a continuous function

$$\arg : D \to \mathbb{R} \quad \text{such that} \quad \operatorname{cis}(\arg(z)) = \frac{z}{|z|}.$$

A *branch of logarithm* is a continuous function

$$\log : D \to \mathbb{C} \quad \text{such that} \quad \exp(\log(z)) = z.$$

Proposition 7. *Let $\arg_1 : D_1 \to \mathbb{R}$ and $\arg_2 : D_2 \to \mathbb{R}$ be branches of argument. Let $D = D_1 \cap D_2$. If D is connected, then there exists an integer $k \in \mathbb{Z}$ such that for all $z \in D$,*

$$\arg_2(z) = \arg_1(z) + 2\pi k.$$

Reason. Let $z \in D$, and let $\theta_1 = \arg_1(z)$, $\theta_2 = \arg_2(z)$. Now $\operatorname{cis}(\theta_1) = \operatorname{cis}(\theta_2)$, so $\theta_2 = \theta_1 + 2\pi k$ for some $k \in \mathbb{Z}$. We need to say why the same k works for all $z \in D$.

Continuous functions map connected sets to connected sets. The difference of continuous functions is continuous. Thus $\dfrac{(\arg_2 - \arg_1)}{2\pi}(D)$ is a connected subset of $\mathbb{Z}$, and so is a point. Thus k must be constant, and does not depend on z. □

Proposition 8. *Let $\log_1 : D_1 \to \mathbb{R}$ and $\log_2 : D_2 \to \mathbb{R}$ be branches of logarithm. Let $D = D_1 \cap D_2$. If D is connected, then there exists an integer $k \in \mathbb{Z}$ such that for all $z \in D$,*

$$\log_2(z) = \log_1(z) + 2\pi k i.$$

Proof. Let $z \in D_1 \cap D_2$. Let $\log_1(z) = x_1 + iy_1$ and $\log_2(z) = x_2 + iy_2$. Then $\exp(x_1 + iy_1) = z = \exp(x_2 + iy_2)$, so $e_1^x \operatorname{cis} y_1 = e_2^x \operatorname{cis} y_2$, which implies that $x_2 = x_1$ and $y_2 = y_1 + 2\pi k$ for some k. Thus $\log_2(z) - \log_1(z) = (x_2 + iy_2) - (x_1 + iy_1) = i(y_2 - y_1) = 2\pi k i$. Since these function are continuous, k must be constant throughout $D_1 \cap D_2$. □

7. Branches of Inverse

Definition 5. Let U be an open subset of $\mathbb{C}$ and let $f : U \to \mathbb{C}$. A *branch of inverse* of f is a continuous function $g : D \to U$, where D is open and connected, such that $f(g(z)) = z$ for every $z \in D$.

It is now our mission to determine the analytic properties of branches of inverse; in particular, if the original function is differentiable, then is its inverse? There are a couple of loose ends we need to tie up before proceeding to the main theorem.

7.1. Loose Ends.

We need to establish a few principles related to nonzero denominators before proceeding to the differentiability of branches of inverse for complex functions.

Recall the foundational limit laws. We have proved most of these in class. The proofs mirror those for real functions.

Theorem 1. *Let* $\lim_{z\to z_0} f(z) = L$ *and* $\lim_{z\to z_0} g(z) = M$. *Then*

- $\lim_{z\to z_0} [f(z) + g(z)] = L + M$;
- $\lim_{z\to z_0} [f(z)g(z)] = LM$;
- $\lim_{z\to z_0} \left[\frac{f(z)}{g(z)}\right] = \frac{L}{M}$, *if* $M \neq 0$.

These laws imply the following.

Proposition 9. *Let* $\lim_{z\to z_0} f(z) = L$ *and* $\lim_{z\to z_0} (f(z)g(z)) = N$. *If* $L \neq 0$, *then* $\lim_{z\to z_0} g(z) = \frac{N}{L}$.

Proof. We wish to write that $g(z) = \frac{f(z)g(z)}{f(z)}$ for z near z_0; however, this requires that f is nonzero in a deleted neighborhood of z_0. But since $f(z) \to L$ as $z \to z_0$, we know that if we select $\epsilon = \frac{|L|}{2}$, then there exists $\delta > 0$ such that $0 < |z - z_0| < \delta$ implies $|f(z) - L| < \frac{|L|}{2}$, so that $|f(z)| > \frac{|L|}{2}$. So f is nonzero in this neighborhood of z_0.

Now from the third limit law,

$$\lim_{z\to z_0} g(z) = \lim_{z\to z_0} \frac{f(z)g(z)}{f(z)} = \frac{N}{L}.$$

□

Proposition 10. *Let* $D \subset \mathbb{C}$, $f : D \to \mathbb{C}$, *and* $z_0 \in D$. *If* f *is differentiable at* z_0 *and* $f'(z_0) \neq 0$, *then there exists a deleted neighborhood* U *of* z_0 *such that for all* $z \in U$, $f(z) \neq f(z_0)$.

Proof. Suppose that $f'(z_0) = d$ and $d \neq 0$. Then $\lim_{z\to z_0} \frac{f(z) - f(z_0)}{z - z_0} = d$. Thus there exists $\delta > 0$ such that $0 < |z - z_0| < \delta$ implies $\left|\frac{f(z) - f(z_0)}{z - z_0} - d\right| < \left|\frac{d}{2}\right|$. This implies that $\left|\frac{f(z) - f(z_0)}{z - z_0}\right| > \left|\frac{d}{2}\right|$, so $|f(z) - f(z_0)| > |z - z_0|\left|\frac{d}{2}\right|$. The quantity on the right is positive, so $f(z) \neq f(z_0)$. □

7.2. **The Main Theorem.** We are now in a position to better understand the proof of the main theorem regarding the differentiability of branches of inverse.

Theorem 2. *Let U be an open subset of $\mathbb{C}$ and let $f : U \to \mathbb{C}$ be differentiable. Let $g : D \to U$ be a branch of inverse of f and let $z_0 \in D$. If $f'(g(z_0)) \neq 0$, then g is differentiable at z_0, and*

$$g'(z_0) = \frac{1}{f'(g(z_0))}.$$

Thus, if $f' \neq 0$ in $g(D)$, then g is analytic in D, and $g'(z) = \dfrac{1}{f'(g(z))}$.

Proof. We begin by pointing out that, since $f'(g(z_0)) \neq 0$, there exists a deleted neighborhood of z_0 such that $g(z) \neq g(z_0)$ for all z in that deleted neighborhood. For z inside such a neighborhood, we note that

$$1 = \frac{z - z_0}{z - z_0} = \frac{f(g(z)) - f(g(z_0))}{z - z_0} = \frac{f(g(z)) - f(g(z_0))}{g(z) - g(z_0)} \cdot \frac{g(z) - g(z_0)}{z - z_0}.$$

Now that limit on the left side as $z \to z_0$ is clearly 1, so the limit on the right side exists. Since g is continuous, $\lim_{z \to z_0} g(z) = g(z_0)$, so

$$\lim_{z \to z_0} \frac{f(g(z)) - f(g(z_0))}{g(z) - g(z_0)} = f'(g(z_0)).$$

Hence we get that

$$\lim_{z \to z_0} \frac{g(z) - g(z_0)}{z - z_0}$$

exists, since $f'(g(z_0)) \neq 0$, and $1 = f'(g(z_0))g'(z_0)$. Thus $g'(z_0) = \dfrac{1}{f'(g(z_0))}$. □

Clearly Log is a branch of inverse of exp. Any branch of the inverse of exp is called a branch of logarithm. We call Log the *principle branch* of logarithm. As a consequence of the theorem above, we have

Corollary 1. ?? *Let $\log : D \to \mathbb{C}$ be a branch of logarithm. Then*

$$\frac{d}{dz} \log(z) = \frac{1}{z}.$$

Proof. Let $f(w) = \exp w$ and $g(z) = \log(z)$. Then $f(g(z)) = z$, $f'(w) = f(w)$, and

$$g'(z) = \frac{1}{f'(g(z))} = \frac{1}{f(g(z))} = \frac{1}{z}.$$

□

8. Instances of Exponential with Base a

We are motivated by the desire to define w^z using
$$w^z = \exp(\log(w^z)) = \exp(z \log(w)).$$
Since we have multiple choices for log, we are confronted with the realization that fixing w and allowing z to vary is distinctly different from fixing z and allowing w to vary.

Definition 6. Let $a \in \mathbb{C}^*$. An *instance of exponential with base* a is a function
$$\exp_a : \mathbb{C} \to \mathbb{C} \quad \text{of the form } \exp_a(z) = \exp(bz),$$
where $\exp(b) = a$.

Proposition 11. *Let* $\exp_a$ *be an instance of exponential with base* a*, given by* $\exp_a(z) = \exp(bz)$*, where* $\exp(b) = a$*. Then*
$$\frac{d}{dz} \exp_a(z) = b \exp_a(z).$$

Proof. This follows from the chain rule. □

Let $\exp_a$ be an instance of exponential with base a, given by $\exp_a(z) = \exp(bz)$, where $\exp(a) = b$. It is important to explore the extent to which this definition depends on the choice of b; to do this, let $b_1, b_2 \in \mathbb{C}$ such that $\exp(b_1) = \exp(b_2) = a$. Write $b_1 = x_1 + iy_1$ and $b_2 = x_2 + iy_2$. Then $e^{x_1} \operatorname{cis} y_1 = e^{x_2} \operatorname{cis} y_2$. This implies that $x_2 = x_1$ and $y_2 = y_1 + 2\pi k$, for some $k \in \mathbb{Z}$. Now if $z = x + iy$, we compute that (for $j = 1, 2$),
$$\exp(b_j z) = e^{xx_j - yy_j} \operatorname{cis}(xy_j + yx_j).$$
Thus
$$\begin{aligned} \exp(b_1 z) &= e^{xx_1 - yy_1} \operatorname{cis}(xy_1 + yx_1) \\ &= e^{xx_2 - yy_2 + 2\pi yk} \operatorname{cis}(xy_2 + yx_2 + 2\pi xk) \\ &= e^{2\pi yk} \exp(b_2 z). \end{aligned}$$
So, the ratio between the instances of exponential with base a depends on the imaginary part of z.

Definition 7. The *principal instance* of $\exp_a$ is
$$\operatorname{Exp}_a : \mathbb{C} \to \mathbb{C} \quad \text{of the form } \operatorname{Exp}_a(z) = \exp(z \operatorname{Log}(a)).$$

Unless otherwise indicated, we assume a^z is given by $a^z = \operatorname{Exp}_a(z)$. Accordingly, we note that
$$e^z = \exp(z).$$

9. Branches of Power with Exponent a

If log is a branch of logarithm whose domain contains a, then $\log(a) = b$, and $\exp_a(z) = \exp(z \log(a))$, as we defined for real functions. Instances of exponential require selection of a single preimage of b under exp. As z varies, we may keep the same value for b. Power functions are more subtle in this regard.

Definition 8. Let $a \in \mathbb{C}^*$ and let D be an open connected subset of $\mathbb{C}$. A *branch of power with exponent* a is a function

$$\mathrm{pow}_a : D \to \mathbb{C} \quad \text{given by } \mathrm{pow}_a(z) = \exp(a \log(z)),$$

where $\log : D \to \mathbb{C}$ is a branch of logarithm.

Proposition 12. *Let* $\mathrm{pow}_a : D \to \mathbb{C}$ *be a branch of power with exponent* $a \neq 0$, *with* $\mathrm{pow}_a(z) = \exp(a \log(z))$ *for some branch of logarithm* $\log : D \to \mathbb{C}$. *Then*

$$\frac{d}{dz} \mathrm{pow}_a(z) = a \, \mathrm{pow}_{a-1}(z),$$

where $\mathrm{pow}_{a-1} : D \to \mathbb{C}$ *is the branch of power with exponent* $a - 1$ *given by* $\mathrm{pow}_{a-1}(z) = \exp((a-1)\log(z))$.

Proof. We have

$$\begin{aligned}
\frac{d}{dz} \mathrm{pow}_a(z) &= \frac{d}{dz} \exp(a \log(z)) && \text{by definition} \\
&= \exp(a \log(z)) \cdot a \cdot \left(\frac{1}{z}\right) && \text{by the Chain Rule} \\
&= a \exp(a \log(z)) \exp\left(\log\left(\frac{1}{z}\right)\right) && \text{by properties of Log} \\
&= a \exp(a \log(z) - \log(z)) && \text{by properties of exp} \\
&= a \exp((a-1)\log(z)) \\
&= a \exp_{a-1}(z)
\end{aligned}$$

□

Although we wish to think of z^w to be a distinct number, we are confronted with the realization that there are choices to be made. Normally, the power function is more useful in deciding how to choose a meaning for z^w.

Definition 9. Let $a \in \mathbb{C}^*$. The *principal branch* of z^a is

$$\mathrm{Pow}_a : \mathbb{C}^+ \to \mathbb{C} \quad \text{given by} \quad \mathrm{Pow}_a(z) = \mathrm{Exp}(a \, \mathrm{Log}(z)).$$

According to this convention, it is clear that

$$\frac{d}{dz} z^a = a z^{a-1}.$$

It should be noted that $\mathrm{Exp}_a(z) = \mathrm{Pow}_z(a)$.

10. Trigonometric Functions

We wish to motivate our definitions for sin and cos. We have discussed why $e^{i\theta} = \text{cis}(\theta) = \cos\theta + i\sin\theta$. Since sin is an odd function, $e^{-i\theta} = \cos\theta - i\sin\theta$. Adding these equations gives $e^{i\theta} + e^{-i\theta} = 2\cos\theta$, so $\cos\theta = \frac{1}{2}(e^{i\theta} + e^{-i\theta})$. Similarly, $\sin\theta = \frac{1}{2i}(e^{i\theta} - e^{-i\theta})$. This motivates the following definition.

Definition 10. The *extended trigonometric functions* are

$$\sin : \mathbb{C} \to \mathbb{C} \quad \text{given by } \sin z = \frac{\exp(iz) - \exp(-iz)}{2i},$$

and

$$\cos : \mathbb{C} \to \mathbb{C} \quad \text{given by } \cos z = \frac{\exp(iz) + \exp(-iz)}{2}.$$

We also define

$$\tan z = \frac{\sin z}{\cos z}, \cot z = \frac{\cos z}{\sin z}, \sec z = \frac{1}{\cos z}, \text{ and } \csc z = \frac{1}{\sin z}.$$

These extensions of the standard trigonometric functions obey the same identities and derivative rules as their real counterparts. They admit have branches of inverse, which may be written using branches of logarithm.

11. Hyperbolic Functions

Definition 11. The *extended hyperbolic functions* are

$$\sinh : \mathbb{C} \to \mathbb{C} \quad \text{given by } \sinh z = \frac{\exp(z) - \exp(-z)}{2},$$

and

$$\cosh : \mathbb{C} \to \mathbb{C} \quad \text{given by } \cosh z = \frac{\exp(z) + \exp(-z)}{2}.$$

We also define

$$\tanh z = \frac{\sinh z}{\cosh z}, \coth z = \frac{\cosh z}{\sinh z}, \text{sech}\, z = \frac{1}{\cosh z}, \text{ and } \text{csch}\, z = \frac{1}{\sinh z}.$$

These functions also satisfy a extensive set of identities, differentiation formulae, and properties of inverse which are extensions of their real counterparts.

12. Exercises

Problem 1. Use Corollary **??** to give an alternate proof of 8:

Let $\log_1 : D_1 \to \mathbb{R}$ and $\log_2 : D_2 \to \mathbb{R}$ be branches of logarithm. Let $D = D_1 \cap D_2$. If D is connected, then there exists an integer $k \in \mathbb{Z}$ such that for all $z \in D$,

$$\log_2(z) = \log_1(z) + 2\pi k i.$$

Solution. Since both are branches of logarithm, $\frac{d}{dz}\log_1 = \frac{d}{dz}\log_2 = \frac{1}{z}$. Since these functions have the same derivative, they differ by a constant, that is, there exists a constant $C \in \mathbb{C}$ such that $\log_2(z) = \log_1(z) + C$ for all $z \in D$. Take exp of both sides to see that

$$z = \exp(\log_2(z)) = \exp(\log_1(z) + C) = z\exp(C).$$

Thus $\exp(C) = 1$, so $e^{\mathrm{Re}(C)}\operatorname{cis}(\mathrm{Im}(C)) = 1$, so $\mathrm{Re}(C) = 0$ and $\mathrm{Im}(C) = 2\pi k$ for some $k \in \mathbb{Z}$. This gives $\log_2(z) = \log_1(z) + 2\pi k i$. $\square$

Problem 2. Let $\arg : D \to \mathbb{C}$ be a branch of argument, and define $\log : D \to \mathbb{C}$ by $\log(z) = \ln|z| + \arg z$. Show that log is a branch of logarithm. Does every branch of logarithm arise in this way?

Problem 3. Let $a \in \mathbb{C}^*$ and let $\exp_a(z) = \exp(bz)$ be an instance of exponential with base a. Find the domain and range of $\exp_a$. Is $\exp_a$ periodic? If so, find its period.

Problem 4. There are infinitely many branches of log. Does each produce a different branch of pow_a? How does the answer to this question depend on a?

Problem 5. We could define e^{iz} in these ways:

- $e^{iz} = \exp(iz)$
- $e^{iz} = \exp(iz\log(e))$

Is there any difference in these definitions? Explain.

Problem 6. Show that, if $z \in \mathbb{R}$, the definition above agrees with the standard definition of sine and cosine on the reals.

Problem 7. Find the domain and range of sine and cosine. Are these functions periodic? If so, find their periods.

Problem 8. Show that $\cos^2 z + \sin^2 z = 1$ for all $z \in \mathbb{C}$.

Problem 9. Show that $\frac{d}{dz}\sin z = \cos z$.

Problem 10. Show that $\frac{d}{dz}\sinh z = \cosh z$ and that $\frac{d}{dz}\cosh z = \sinh z$.

Problem 11. Show that $\cosh^2 z - \sinh^2 z = 1$.

Problem 12. Set $w = \cos z$; then

$$w = \frac{1}{2}(e^{iz} + e^{-iz}).$$

Solve this for e^{iz}, using the quadratic formula. Then come up with a formula of $\arccos z$.

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